Homological algebra exercise sheet Week 9

1. In this exercise we will show the snake lemma using the notion of convergence of first quadrant spectral sequences. Recall:

   Lemma 0.1 (Snake Lemma)**.** *Consider a commutative diagram in an abelian category $\mathcal{A}$*

$$\begin{tikzcd}
 & A' \arrow[r, "\alpha"] \arrow[d, "f"] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r, "\beta"'] & C &
\end{tikzcd}$$

   such that the rows are exact. Then there is an exact sequence

$$\ker f \to \ker g \to \ker h \to \textit{coker } f \to \textit{coker } g \to \textit{coker } h$$

   Moreover if α is monic, then so is $\ker f \to \ker g$, and if β is epi, then so is coker $g \to$ coker h.

   For this, consider the first quadrant double complex with bottom left corner being

$$\begin{tikzcd}
0 & C' \arrow[l] \arrow[d, "h"] & B' \arrow[l] \arrow[d, "g"] & A' \arrow[l, "\alpha"'] \arrow[d, "f"] & K' \arrow[l] \\
K & C \arrow[l] & B \arrow[l, "\beta"] & A \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

   where $K' = \ker \alpha, K = \text{coker } \beta$, and define the associated spectral sequence as in Exercise 1 of Sheet 8. Throughout this exercise, we will assume the true fact that this spectral sequence converges to the homology of the total complex $H_*(T)$ (see for example Section 5.6 of Weibel's book).

   (a) Show that every term of the E^2 page is 0 except for $E^2_{3,0}$ and $E^2_{1,1}$ which are isomorphic one to each other.

   (b) Show the snake lemma.

2. Let $\{E^r_{pq}\}$ be a regular upper half-plane spectral sequence, that is $E^r_{pq} = 0$ whenever $q < 0$, such that for any p, q with $q \geq 0$, $E^\infty_{pq} \cong \mathbb{Z}/2\mathbb{Z}$. Show that E approaches H_* and converges to H'_*, where $H_n := \mathbb{Z}, H'_n := \mathbb{Z}_2$ (the 2-adic numbers) for any n.

3. In this exercise we will construct the Wang sequence. Suppose that we have a fibration

$$F \xrightarrow{i} E \xrightarrow{\pi} S^n$$

where the base space is an n-sphere for $n \geq 2$.

(a) Show that we have an exact sequence

$$0 \to E^{\infty}_{n,q} \to H_q(F) \xrightarrow{d^n} H_{q+n-1}(F) \to E^{\infty}_{0,q+n-1} \to 0$$

for each $q \geq 0$, where d^n is a differential from the Leray-Serre spectral sequence.

(b) Show that for each $q \geq 0$ we have a short exact sequence

$$0 \longrightarrow E^{\infty}_{0,q} \longrightarrow H_q(E) \longrightarrow E^{\infty}_{n,q-n} \longrightarrow 0.$$

(c) Use the last two exercises to show that there is a long exact sequence

$$... \to H_q(F) \xrightarrow{i_\star} H_q(E) \to H_{q-n}(F) \xrightarrow{d^n} H_{q-1}(F) \xrightarrow{i_\star} H_{q-1}(E) \to ...$$

4. In this exercise we will calculate the homology groups of the loop space ΩS^n for $n \geq 2$.

(a) Use the Serre long exact sequence to calculate $H_p(\Omega S^n)$ for $p \leq n-1$.

(b) Use induction and the Wang sequence to calculate $H_p(\Omega S^n)$ for $p \geq n$.